$$h_{j_1, \ldots, j_s} = \frac{s!}{1^{j_1} j_1! \, 2^{j_2} j_2! \ldots s^{j_s} j_s!}.$$

7. For free Husimi trees all points belong to the same set. The dissimilarity characteristic relation (equation (6)) is however also valid in the more general case.

8. Harary, F., and Norman, R. Z., "The Dissimilarity Characteristic of Husimi Trees," *Annals Math.*; (in process of publication).

9. We will make special use of the result (Pólya, Ref. 5, pp. 161 and 172) that the number of ways in which n *different* figures, from the figure collection counted by $f(x)$, may be distributed on n points is counted by

$$Z(\mathfrak{A}_n, f(x)) - Z(\mathfrak{S}_n, f(x)),$$

where $\mathfrak{A}_n$ is the alternating group of degree n.

We also make frequent use of Pólya's result, p. 164, that for two independent figure collections, the corresponding counting series is the product of the individual ones.

10. In this and all subsequent formulas for the $Y_t(x)$, the extra factor x is needed to account for the cycle to which all the rooted trees, A, B, ... are connected.

STUDIES IN THE CONFORMAL MAPPING OF RIEMANN SURFACES, I

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1. We are concerned with an analysis of the exact form of the Lindelöf inequality for directly conformal maps of one Riemann surface with positive boundary into another. Let F and G denote Riemann surfaces with positive boundary, let $\mathcal{G}_F$ and $\mathcal{G}_G$ denote their respective Green's functions, let f denote a directly conformal map of F into G (not necessarily univalent onto), and let $n(p)$ denote the multiplicity of f at $p \in F$. The exact form of the Lindelöf inequality may be taken as the assertion that for such F, G, f for each $q \in G$, the residual term u_q in

$$\mathcal{G}_G[f(p), q] = \sum_{f(r)=q} n(r)\mathcal{G}_F(p, r) + u_q(p) \qquad (1.1)$$

is non-negative harmonic on F. We observe that u_q is the greatest harmonic minorant of $\mathcal{G}_G[f(p), q]$.

We shall use the following terminology.[2] A non-negative harmonic function on a Riemann surface will be termed *quasi-bounded* provided that it may be represented as the limit of a monotone non-decreasing sequence of non-negative bounded harmonic functions all having the given surface as their domain; a non-negative harmonic function on a Riemann surface will be termed *singular* provided that the only bounded non-negative harmonic function on the surface dominated by it is identically zero. Parreau[2] has shown that a non-negative harmonic function on a Riemann surface admits a unique representation as the sum of a quasi-bounded harmonic function and a singular harmonic function having the given surface as their domain. Our first result is:

THEOREM A: (1) *$(p, q) \to u_q(p)$ is upper semi-continuous on $F \times G$. Let v_q denote the quasi-bounded component of u_q. Then $(p, q) \to v_q(p)$ is lower semi-continuous on $F \times G$. (2) Either $v_q = 0$ for all $q \in G$ or else for no q. (3) The set of q for which $u_q - v_q > 0$ is an F_σ of capacity zero.*

Part (3) generalizes in several respects a corresponding result proved by Frostman[1] for Blaschke products. The proof of (1) depends upon the upper semicontinuity of $q \to u_q(p)$ for each p and the lower semicontinuity of $q \to v_q(p)$ as well as uniformity properties of non-negative harmonic functions that are assured by the Harnack inequality. (2) depends upon known properties of the Green's function. The proof of (3) lies deeper as might be expected. The key idea of the proof lies in showing that for a Green's potential P on G, where

$$P(q) = \int \mathcal{G}_G(q, s)d\mu(s), \tag{1.2}$$

μ being a non-negative mass distribution on G, with compact kernel, the greatest harmonic minorant of $P \circ f$ is

$$\int u_q d\mu(q) \tag{1.3}$$

and the quasi-bounded component of (1.3) is

$$\int v_q d\mu(q) \tag{1.4}$$

2. *Maps of Type-Bl.* We term f of *type*-Bl provided that v_q vanishes (for all q). In case $F = G = (|z| < 1)$, this is tantamount to saying that for $|\alpha| < 1$,

$$\frac{f - \alpha}{1 - \bar{\alpha}f}$$

is a Blaschke product save for a set of α of capacity zero.

We say that f is of *type-Bl at $q(\in G)$* provided that there exists a relatively compact Jordan region Ω, $q \in \Omega \subset G$ such that (1) $f^{-1}(\Omega)$ is not empty, (2) the restriction of f to a component of $f^{-1}(\Omega)$ is of type-Bl relative to Ω for *each* component of $f^{-1}(\Omega)$. We say that f is *locally of type-Bl* provided that f is of type-Bl at each point of G. It is to be remarked that in these latter definitions the hypothesis that F and G have positive ideal boundaries may be dropped. We have:

THEOREM B: (1) *If G has (as previously) positive ideal boundary and f is of type-Bl, then for each region $\Omega \subset G$, the restriction of f to a component of $f^{-1}(\Omega)$ is of type Bl relative to Ω for each component of $f^{-1}(\Omega)$.* (2) *If G has positive ideal boundary and f is locally of type-Bl, then f is of type-Bl.*

An essential component of the proof is the following lemma.

LEMMA C. *Given u subharmonic on F and f locally of type-Bl, let U be defined on G as the upper function of* $\sup_{f(p)=q} u(p)$. *If* $U < +\infty$, *then U is subharmonic on G.*

A consequence of theorem B is that the classical theorem of Iversen may be extended to maps which are locally of type-Bl. Lemma C admits a number of applications to other problems. By way of illustration we cite:

THEOREM D: *Let G have positive ideal boundary and let u be singular positive harmonic on G. If* $f:F \rightarrow G$ *is of type-Bl, then* $u \circ f$ *is singular on F.*

Theorem D leads to the following consequence for convergent infinite Blaschke products: the set of Fatou boundary values of modulus one of a convergent infinite Blaschke product is the unit circumference and each such Fatou boundary value of modulus one is attained at infinitely many points of the unit circumference.

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[1] Frostman, O., Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. Lund Thesis (1935).

[2] Parreau, M., Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann. Paris Thesis (1952).

THE PARAMETRISATION AND ELEMENT OF VOLUME OF THE UNITARY SYMPLECTIC GROUP

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We have given previously in these PROCEEDINGS[1] a system of parameters for, and the complete element of volume (including the in-class factor as well as the well-known class factor) of, the n-dimensional rotation and unitary groups and we complete, in the present note, this problem for the classical groups by giving a system of parameters for, and the complete element of volume of, the $2k$-dimensional unitary symplectic group.

The $2k$-dimensional symplectic group, over the field of complex numbers, may be presented as the collection of $2k \times 2k$ matrices, X, with complex elements, which are such that $X'I_{2k}X = I_{2k}$ where $I_{2k} = \begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix}$. Writing X as a 2×2 block matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$, whose elements are $k \times k$ ma-